Math for Economists, Winter 2023

List 1

Absolute value, inequalities, polynomials

1. Re-write the expressions below as either numbers or piecewise functions (do not use absolute value notation).

   a) $|1-\sqrt{3}|$, b) $|x+x^2|$, c) $x+|1-x|+2|x-2|$,

   d) $|3x-8|$, e) $|x+1|-x$, f) $|x-1|+\frac{x}{|x|}-|x+2|$.

2. Using the geometric meaning of the absolute value, draw the sets of points satisfying conditions below. Write down the solutions of equations or inequalities.

   a) $|x+4|=2$, b) $|3x-2|>1$, c) $|6-2x|\leq 3$,

   d) $|x+2|=|3-x|$, e) $|x+3|>|x-1|$, f) $|x|+|x-\sqrt{6}|=1$,

   g) $|x+1|+|x-2|=3$, h) $|x-5|+|x|<5$, i) $|x+1|+|x-3|>4$.

3. Write down the sets below using the absolute value notation $|\cdot|$.

   a) $\{4,18\}$, b) $\{1+\sqrt{3},3+\sqrt{3}\}$, c) $-3<x<3$,

   d) $0\leq x\leq 2\sqrt{5}$, e) $x\in(-\infty,4)\cup(10,+\infty)$, f) $x\in(-\infty,-\sqrt{2}]\cup[2+\sqrt{2},+\infty)$.

4. Solve equations and inequalities

   a) $|x|+\sqrt{2}=|x+\sqrt{2}|$, b) $|x+1|+|x-2|=5$, c) $|3x+1|=|3-x|$,

   d) $|x-2|<x$, e) $|3-3x|\geq 6-3x$, f) $|1-2x|-|x+3|>x+4$.

5. Write the quadratic functions below in the product form (if it exists) and in the canonical form; draw the graphs:

   a) $-x^2+x$, b) $2x^2+1$, c) x^2+2x-3,

   d) $x^2+x+\frac{1}{4}$, e) $-2x^2-2x+\frac{3}{2}$, f) $-x^2-3x-\frac{9}{4}$.

6. Determine the values of the parameter m such that the function
   $$f(x)=(m-3)x^2+(m-3)x+m-2$$
   a) is linear. Draw the graph of $f(x)$;
   b) is a quadratic function with one root. Draw the graph of $f(x)$,
   c) the largest value of $f(x)$ is positive.

7. What are the values of m such that the function $f(x)=mx^2+4x+m-3$:
   a) has a root,
   b) has two roots of different signs,
   c) has two positive roots,
   d) has the smallest value and this value is positive.

8. Let $g(m)$ be the number of intersection points of a line $y=mx-3$ and the graph of $y=(m+1)x^2+(2-m)x-2$, depending on m.
   Draw the graph of $g(m)$.

9. Find the coefficients and determine the degree of polynomials:

   a) $(x^4-3x^3+x-1)(x^2-x+4)$, b) $y=(x^3+5x^2-x+3)(x-2)^2$,

   c) $W(x)=(x+2)^3-(x-1)^2$, d) $y=(x+1)^2-(2x+3)^3-2x$.

10. Calculate the quotient and the remainder in the division of P by Q:
a) $P(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$, $Q(x) = x^2 - 3x + 1$,
b) $P(x) = x^{16} - 16$, $Q(x) = x^4 + 2$,
c) $P(x) = x^5 - x^3 + 1$, $Q(x) = (x-1)^3$.

11. Find the value of a such that the remainder in the division of $W(x) = 2x^3 + (a^2+1)x^2 - (a+2)x - 6$ by $Q(x) = x + 3$ is as small as possible.

12. Find all integer roots of the polynomials:
a) $x^3 + x^2 - 4x - 4$, b) $3x^3 - 7x^2 + 4x - 4$,
c) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$, d) $x^4 + 3x^3 - x^2 + 17x + 99$.

13. Find all rational roots of the polynomials:
a) $4x^3 + x - 1$, b) $3x^4 - 8x^3 + 6x^2 - 1$,
c) $x^3 - \frac{7}{6}x^2 - \frac{3}{2}x - \frac{1}{3}$, d) $x^5 + \frac{4}{3}x^3 - x^2 + \frac{1}{3}x - \frac{1}{3}$.

14. Write the polynomials below as products of irreducible components:
a) $x^6 + 8$, b) $x^4 + x^2 + 1$, c) $x^4 - x^2 + 1$,
d) $4x^5 - 4x^4 - 13x^3 + 13x^2 + 9x - 9$.

15. Solve the equations:
a) $x^3 - 3x - 2 = 0$, b) $3x^4 - 10x^3 + 10x - 3 = 0$,
c) $x^6 - 2\sqrt{2}x^3 + 2 = 0$, d) $x^4 - 2x^2 + 3x - 2 = 0$.

16. Solve the inequalities:
a) $x^3 - x^2 + 4x < 4$, b) $x^3 - 6x^2 + 5x + 12 > 0$,
c) $(1 - x^2)(4x^2 + 8x - 21) \geq 0$, d) $x^4 + 3x^3 + x^2 \leq 0$.

17. Solve:

a) $\dfrac{12}{1-9x^2} = \dfrac{1-3x}{1+3x} + \dfrac{1+3x}{3x-1}$, b) $\dfrac{30}{x^2-1} - \dfrac{13}{1+x+x^2} = \dfrac{7+18x}{x^3-1}$,

c) $\dfrac{5}{x^2-4} + \dfrac{18}{x^2-3x+2} = \dfrac{8}{x^2-1}$, d) $\dfrac{x}{x+a} + \dfrac{x}{x-a} = \dfrac{8}{3}$.

18. Solve the inequalities:

a) $\dfrac{(x-1)^2}{(x+1)^3} \leq 0$, b) $\dfrac{x^2+2}{x+1} < 2$, c) $2 + \dfrac{3}{x+1} > \dfrac{2}{x}$,

d) $\dfrac{1}{(x+1)^3} > \dfrac{1}{x+1}$, e) $\dfrac{x^2-5}{x} < x + 1$, f) $\left|\dfrac{2x-3}{x-1}\right| \geq 2$,

g) $\left|\dfrac{x^2-5x+3}{x^2-1}\right| < 1$, h) $\dfrac{x}{|x-2|} < 3$, i) $\dfrac{\sqrt{x^2+6x+9}}{x} \geq -2$.

19. For which values of the parameters a and b does the equation

$$a + \frac{b}{x} = \frac{x-2}{x}$$

have a solution (for x)? Also answer this question for the equation

$$1 + \frac{b}{x} = \frac{x}{x-a}.$$

20. Prove that no integer can satisfy the inequality

$$\frac{1}{x} + \frac{1}{x+1} < \frac{2}{x+2}.$$

21. Draw the graphs of functions:

a) $f(x) = |6 - 2x|$, b) $f(x) = 6 - |x|$,

c) $f(x) = \sqrt{x^2 - 6x + 9} + |x|$, d) $f(x) = x^2 - |x| + 1$,

e) $f(x) = (2x - 3)/(x + 1)$, f) $f(x) = \mathrm{sgn}(x^2 - 3x)$.

Note: the function $\mathrm{sgn}(x)$ (the *sign* of x) takes the value $+1$ for $x > 0$, 0 for $x = 0$, and -1 for $x < 0$.